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Deconstructing plane anisotropic elasticity Part I: The latent structure of Lekhnitskii's formalism

Wan-Lee Yin*

School of Civil and Environmental Engineering, Georgia Institute of Technology, Atlanta, GA 30332-0355, USA

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Abstract

General solutions of the stress and displacements in two-dimensional anisotropic elasticity may be represented by eigenvectors and analytic functions of the complex variables $x + \mu_i y$, but the representation takes different forms for five distinct types of materials as determined by the elastic compliance matrix $[\beta]$. In this paper, explicit expressions of the general solutions are derived for each type of anisotropic materials in terms of the eigenvalues μ_i and the elements of $[\beta]$. It is shown that, for degenerate and extra-degenerate materials, the generalized eigenvectors and associated eigensolutions may be obtained by the *derivative rule*. The Barnett–Lothe tensors are defined in terms of *unnormalized* eigenvectors by the same set of relations regardless of material degeneracy. Explicit expressions of these tensors are given in concise forms depending only on the multiplicity of the eigenvalues. The six-dimensional matrix formalism and normalization of the eigenvectors are found to be neither essential nor expedient for the analysis except as a device for abridged expressions of matrix identities.

Keywords: Anisotropic elasticity; Lekhnitskii's formalism; Stroh's formalism; Barnett–Lothe tensors; Degenerate materials

1. Introduction

Two-dimensional fundamental solutions of anisotropic elastic bodies were presented first by Lekhnitskii (1963) using a compliance-based formalism, and later by Stroh, 1958 and others in terms of the anisotropic moduli of elasticity. It is well-known that the usual representation of the fundamental solutions breaks down in the case of degenerate and extra-degenerate materials — materials with fewer than three complex conjugate pairs of eigenvectors for the stresses or the displacements (see Ting (1996) for a general introduction and references to anisotropic elasticity). This includes the important class of

* Tel.: +1 404 894 2733; fax: +1 404 894 2278.
E-mail address: wanlee.yin@ce.gatech.edu (W.-L. Yin).

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isotropic materials. More recently, Ting and Hwu (1988) showed the eigensolutions of materials that are degenerate but not extra-degenerate. The case of extra-degenerate materials (with a triple eigenvalue that has only one independent eigenvector) has been scarcely explored (Wang and Ting, 1997), and even the existence of such materials had been questioned until quite recently.

In the present paper, explicit expressions of fundamental solutions are given for all types of anisotropic materials, whether nondegenerate, degenerate or extra-degenerate. For each type of material, the eigenvectors and generalized eigenvectors form a linearly independent system and satisfy modified orthogonality and closure relations. By using these relations, explicit expressions are obtained for the Barnett–Lothe tensors in terms of the eigenvalues and the anisotropic compliancies. The Barnett–Lothe tensors are here defined by the same expressions (Eqs. (3.2) and (3.4) of this paper) regardless of material degeneracy, in terms of *unnormalized* eigenvectors.

A crucial relation that makes the structure of plane anisotropic elastostatic solutions so tangible is that the first two components of the **b**-vector (the eigenvector of the stress potentials) corresponding to an eigenvalue μ have the ratio $-\mu$. This relation, resulting from the existence of the Airy stress function, introduces an asymmetry in the dual formalism, since the **a**-vector associated with the displacement field has no comparable property. Consequently, the expressions of the eigenvectors and eigensolutions of various material types are generally simpler and more explicit in a compliance-based (Lekhnitskii) formalism than in a stiffness-based (Stroh) formalism.

In the degenerate and extra-degenerate cases, the generalized eigenvectors may be obtained by the *derivative rule*. First, analytical expressions of the eigenvectors **a** and **b** are obtained as polynomial functions of μ *before evaluating* μ at the repeated root μ_0. Differentiation of the expressions with respect to μ and subsequent evaluation at $\mu = \mu_0$ then yields the generalized eigenvectors. For extra-degenerate materials, repeated differentiation and subsequent evaluation at $\mu = \mu_0$ gives a second set of generalized eigenvectors. Generalized eigensolutions for the displacements, stress potentials, stress and strain may also be obtained by using the same rule.

Our investigation leads naturally to five distinctive types of anisotropic materials, each having different representations of the displacement and stress solutions. These material types are determined by the *multiplicity* of eigenvalues and, in the case of a multiple root μ_0, *whether or not* the eigenmatrix $\mathbf{M}(\mu_0)$ or $\boldsymbol{\Gamma}(\mu_0)$ *has a vanishing adjoint matrix* (see Eqs. (2.2b) and (2.12) in the next section for the definitions of $\mathbf{M}(\mu)$ and $\boldsymbol{\Gamma}(\mu_0)$). If $\mathbf{M}(\mu)$ and, therefore, its adjoint matrix does not vanish for any real or complex number μ, then each eigenvalue, whether a simple or multiple root, is associated with only one independent eigenvector. Such materials will be called *normal* (as will be shown in Part II of this paper, their 6×6 eigenmatrix **N** are either simple or non-semisimple). If $\mathbf{M}(\mu_0) = \mathbf{0}$ for some μ_0, then there are exactly two independent eigenvectors associated with the double or triple root μ_0, and the material will be called *abnormal*. This characterization implies, in particular, that there is no need to give a separate analysis and classification for the "M_3 materials" (materials such that the matrix function $\mathbf{M}(\mu)$ is always diagonal). Although abnormal materials are pathological in a mathematical sense, they are more familiar in the common sense because isotropic materials and materials transversely isotropic in the x–y plane are both abnormal.

A parallel investigation based on the Stroh formalism leads to the same classification of materials, but the eigensolutions and the Barnett–Lothe tensors are all expressed in terms of the eigenvalues and the anisotropic elastic moduli. The latter set of expressions are derived in Part II of this paper. The dualism and asymmetry of the two formalisms are made transparent in this study.

2. Five types of anisotropic elastic materials

Let α_{ij} $(i, j = 1, \ldots, 6)$ denote the anisotropic elastic compliance constants relating the strain components ϵ_x, ϵ_y, ϵ_z, γ_{yz}, γ_{xz}, γ_{xy} to the stress components σ_x, σ_y, σ_z, τ_{yz}, τ_{xz}, τ_{xy}, and let

$$\beta_{ij} = \alpha_{ij} - \alpha_{i3}\alpha_{j3}/\alpha_{33} \quad (\text{for } i, j \neq 3).$$

Then, for generalized plane deformations, one has (Lekhnitskii, 1963)

$$\{\epsilon\} = [\beta]\{\sigma\}, \tag{2.1}$$

where $\{\epsilon\} = \{\epsilon_x, \epsilon_y, \gamma_{yz}, \gamma_{xz}, \gamma_{xy}\}^{\mathrm{T}}$, $\{\sigma\} = \{\sigma_x, \sigma_y, \tau_{yz}, \tau_{xz}, \tau_{xy}\}^{\mathrm{T}}$ and

$$[\beta] = \begin{bmatrix} \beta_{11} & \beta_{12} & \beta_{14} & \beta_{15} & \beta_{16} \\ \beta_{12} & \beta_{22} & \beta_{24} & \beta_{25} & \beta_{26} \\ \beta_{14} & \beta_{24} & \beta_{44} & \beta_{45} & \beta_{46} \\ \beta_{15} & \beta_{25} & \beta_{45} & \beta_{55} & \beta_{56} \\ \beta_{16} & \beta_{26} & \beta_{46} & \beta_{56} & \beta_{66} \end{bmatrix}$$

We define the matrix functions

$$\mathbf{P}(\mu) \equiv \begin{bmatrix} -\mu^2 & 0 \\ -1 & 0 \\ 0 & -1 \\ 0 & \mu \\ \mu & 0 \end{bmatrix}, \tag{2.2a}$$

$$\mathbf{M}(\mu) \equiv \mathbf{P}^{T}(\mu)[\beta]\mathbf{P}(\mu) = \begin{bmatrix} l_4(\mu) & -l_3(\mu) \\ -l_3(\mu) & l_2(\mu) \end{bmatrix}. \tag{2.2b}$$

where $\mathbf{P}^{\mathrm{T}}$ denotes the transpose of $\mathbf{P}$. The characteristic equation

$$\delta(\mu) \equiv |\mathbf{M}(\mu)| = l_2(\mu)l_4(\mu) - l_3(\mu)^2 = 0 \tag{2.3}$$

has three pairs of complex conjugate roots $\{\mu_k, \bar{\mu}_k\}$ ($k = 1, 2, 3$). Lekhnitskii (1963) presented the general form of plane elastostatic solutions of anisotropic media in terms of the stress functions $F^{(k)}(x + \mu_k y)$ and $\Psi^{(k)}(x + \mu_k y)$ and their complex conjugates, assuming that the μ_k's are all distinct.

In the absence of body forces, the equilibrium conditions imply that σ may be represented by the derivatives of a pair of stress functions $F(x, y)$ and $\Psi(x, y)$:

$$\sigma_x = F_{,yy}, \quad \sigma_y = F_{,xx}, \quad \tau_{xy} = -F_{,xy},$$

$$\tau_{xz} = \Psi_{,y}, \quad \tau_{yz} = -\Psi_{,y},$$

Solutions for the displacements $\mathbf{u} \equiv \{u, \ v, \ w\}^{\mathrm{T}}$ and the stress potentials $\mathbf{q} \equiv \{F_{,y}, -F_{,x}, \Psi\}^{\mathrm{T}}$ may be sought in the form

$$\mathbf{u} = \sum \mathbf{a}^{(i)} f_i(x + \mu_i y), \tag{2.4a}$$

$$\mathbf{q} = \sum \mathbf{b}^{(i)} f_i(x + \mu_i y) \tag{2.4b}$$

or,

$$\begin{Bmatrix} u \\ v \\ w \end{Bmatrix} = \sum \begin{Bmatrix} a_1^{(i)} \\ a_2^{(i)} \\ a_3^{(i)} \end{Bmatrix} f_i(x+\mu_i y), \quad \begin{Bmatrix} F_{,y} \\ -F_{,x} \\ \Psi \end{Bmatrix} = \sum \begin{Bmatrix} b_1^{(i)} \\ b_2^{(i)} \\ b_3^{(i)} \end{Bmatrix} f_i(x+\mu_i y).$$

where μ_i's ($i = 1, 2, \ldots, 6$) are complex constants to be determined and f_i's are arbitrary complex-valued analytic functions. Differentiating Eqs. (2.4a) and (2.4b), one obtains the strain and stress components

$$\{\epsilon\} = \begin{Bmatrix} \epsilon_x \\ \epsilon_y \\ \gamma_{yz} \\ \gamma_{xz} \\ \gamma_{xy} \end{Bmatrix} = \begin{bmatrix} \partial_x & 0 & 0 \\ 0 & \partial_y & 0 \\ 0 & 0 & \partial_y \\ 0 & 0 & \partial_x \\ \partial_y & \partial_x & 0 \end{bmatrix} \begin{Bmatrix} u \\ v \\ w \end{Bmatrix} = \sum \begin{bmatrix} 1 & 0 & 0 \\ 0 & \mu_i & 0 \\ 0 & 0 & \mu_i \\ 0 & 0 & 1 \\ \mu_i & 1 & 0 \end{bmatrix} \begin{Bmatrix} a_1^{(i)} \\ a_2(i) \\ a_3(i) \end{Bmatrix} f_i'(x+\mu_i y) \tag{2.5a}$$

$$\{\sigma\} = \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{yz} \\ \tau_{xz} \\ \tau_{xy} \end{Bmatrix} = \begin{bmatrix} \partial_y & 0 & 0 \\ 0 & -\partial_x & 0 \\ 0 & 0 & -\partial_x \\ 0 & 0 & \partial_y \\ -\partial_x & 0 & 0 \end{bmatrix} \begin{Bmatrix} F_{,y} \\ -F_{,x} \\ \Psi \end{Bmatrix} = \sum \begin{bmatrix} \mu_i & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & \mu_i \\ -1 & 0 & 0 \end{bmatrix} \begin{Bmatrix} b_1^{(i)} \\ b_2(i) \\ b_3(i) \end{Bmatrix} f_i'(x+\mu_i y) \tag{2.5b}$$

Since

$$\tau_{xy} = -\partial_x F_{,y} = -\sum b_1^{(i)} f_i'(x+\mu_i y) = \partial_y\left(-F_{,x}\right) = \sum b_2^{(i)} \mu_i f_i'(x+\mu_i y)$$

one has

$$b_1^{(i)} = -\mu_i b_2^{(i)} \tag{2.6}$$

It follows that

$$\{\sigma\} = \sum \mathbf{P}(\mu_i) \begin{Bmatrix} b_2^{(i)} \\ b_3^{(i)} \end{Bmatrix} f_i'(x+\mu_i y), \tag{2.7}$$

where $\mathbf{P}(\mu)$ was defined in Eq. (2.2a). We define additional matrix functions

$$\mathbf{E}(\mu) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & \mu \\ 0 & 0 & 1 \\ \mu & 1 & 0 \end{bmatrix} \tag{2.8a}$$

$$\mathbf{Y}(\mu) = \begin{bmatrix} 1 & -\mu & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \tag{2.8b}$$

Then,

$$\mathbf{Y}^{\mathrm{T}}\mathbf{E} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{2.9a}$$

$$\mathbf{E}^{\mathrm{T}}\mathbf{P} = \mathbf{P}^{\mathrm{T}}\mathbf{E} = \mathbf{0}, \tag{2.9b}$$

The elements of the matrix function $\mathbf{M}(\mu)$ of Eq. (2.2b) are given by

$$l_4(\mu) = \beta_{11}\mu^4 - 2\beta_{16}\mu^3 + (2\beta_{12} + \beta_{66})\mu^2 - 2\beta_{26}\mu + \beta_{22}$$

$$l_3(\mu) = \beta_{15}\mu^3 - (\beta_{14} + \beta_{56})\mu^2 + (\beta_{25} + \beta_{46})\mu - \beta_{24}$$

$$l_2(\mu) = \beta_{55}\mu^2 - 2\beta_{45}\mu + \beta_{44} \tag{2.10}$$

Eqs. (2.1), (2.5a) and (2.7) yield

$$\mathbf{E}(\mu_i)\mathbf{a}^{(i)} = [\beta]\mathbf{P}(\mu_i)\begin{Bmatrix} b_2^{(i)} \\ b_3^{(i)} \end{Bmatrix} \tag{2.11}$$

Pre-multiplication of the last equation by $\mathbf{E}^{\mathrm{T}}(\mu_i)[\beta]^{-1}$ and $\mathbf{P}(\mu_i)^{\mathrm{T}}$ yield, respectively,

$$\mathbf{E}^{\mathrm{T}}(\mu_i)[\beta]^{-1}\mathbf{E}(\mu_i)\mathbf{a}^{(i)} \equiv \boldsymbol{\Gamma}(\mu_i)\mathbf{a}^{(i)} = \mathbf{0}, \tag{2.12}$$

$$\mathbf{P}^{\mathrm{T}}(\mu_i)[\beta]\mathbf{P}(\mu_i)\begin{Bmatrix} b_2^{(i)} \\ b_3^{(i)} \end{Bmatrix} = \mathbf{M}(\mu_i)\begin{Bmatrix} b_2^{(i)} \\ b_3^{(i)} \end{Bmatrix} = 0. \tag{2.13}$$

The last two equations are formally analogous except for the difference in the dimensionality. The relative analytical simplicity of the compliance-based formalism (in terms of β_{ij}) now becomes evident. The present formulation, including the eigenrelation of Eq. (2.13), has been used previously in a general analysis of mechanical and thermal stresses in multi-material wedges (Yin, 1997). We now adopt this analysis approach to obtain the complete set of eigensolutions for the various classes of anisotropic materials. The analysis and results of the alternative approach, based on the well-known eigenrelation of Eq. (2.12), will be given in Part II of this paper.

Eq. (2.2b) yields

$$\mathbf{M}(\mu)\begin{Bmatrix} l_2(\mu) \\ l_3(\mu) \end{Bmatrix} = \delta(\mu)\begin{Bmatrix} 1 \\ 0 \end{Bmatrix} \tag{2.14}$$

$$\mathbf{M}(\mu)\begin{Bmatrix} \sqrt{l_2(\mu)} \\ \sqrt{l_4(\mu)} \end{Bmatrix} = \frac{\delta(\mu)}{l_3 + \sqrt{l_2 l_4}}\begin{Bmatrix} \sqrt{l_4(\mu)} \\ \sqrt{l_2(\mu)} \end{Bmatrix} \tag{2.15}$$

where the square roots $\sqrt{l_2(\mu)}$ and $\sqrt{l_4(\mu)}$ are chosen as follows so as to be consistent with Eq. (2.3):

$$\arg\left(\sqrt{l_2}\right) + \arg\left(\sqrt{l_4}\right) = \arg(l_3).$$

For a *normal* material, the elements $l_2(\mu)$, $l_3(\mu)$ and $l_4(\mu)$ of the matrix $\mathbf{M}(\mu)$ have no common roots. Hence, the singular matrix $\mathbf{M}(\mu_i)$ does not vanish and it must be of rank one. Consequently, Eq. (2.13) has one independent solution vector and it may be chosen as

$$\begin{Bmatrix} b_2^{(i)} \\ b_3^{(i)} \end{Bmatrix} = \begin{Bmatrix} \sqrt{l_2(\mu_i)} \\ \sqrt{l_4(\mu_i)} \end{Bmatrix} \tag{2.16}$$

Notice that this vector is non-trivial because otherwise Eq. (2.3) would imply $l_3(\mu_i) = 0$, so that the material would be *abnormal*. Eqs. (2.6) and (2.11) now yield

$$\mathbf{b}^{(i)} = \mathbf{J}(\mu_i) \begin{Bmatrix} b_2^{(i)} \\ b_3^{(i)} \end{Bmatrix}, \tag{2.17a}$$

$$\mathbf{a}^{(i)} = \mathbf{K}(\mu_i) \begin{Bmatrix} b_2^{(i)} \\ b_3^{(i)} \end{Bmatrix} \tag{2.17b}$$

where

$$\mathbf{J}(\mu) \equiv \begin{bmatrix} -\mu & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}, \tag{2.18a}$$

$$\mathbf{K}(\mu) \equiv \mathbf{Y}^{\mathrm{T}}(\mu)[\beta]\mathbf{P}(\mu) \tag{2.18b}$$

For an *abnormal* material, one has $\mathbf{M}(\mu_0) = \mathbf{0}$ for some eigenvalue μ_0. Two independent pairs of **a**- and **b**-vectors are associated with this eigenvalue, which must be a repeated root of $\delta(\mu) = 0$. Choosing the column vectors of the 2×2 identity matrix to be the independent solutions of Eq. (2.13), we then have

$$\{\mathbf{b}^{(1)}, \mathbf{b}^{(2)}\} = \mathbf{J}(\mu_0), \qquad \{\mathbf{a}^{(1)}, \mathbf{a}^{(2)}\} = \mathbf{K}(\mu_0). \tag{2.19}$$

Thus, the number of independent eigenvectors depends on the multiplicity of the eigenvalues and on whether the material is normal or abnormal. It follows that all anisotropic elastic materials may be classified into five distinct types:

(N-Simple) Normal materials with three simple eigenvalues (The SP group);
(N-Double) Normal materials with one simple and one double eigenvalue (The D1 group);
(N-Triple) Normal materials with one triple eigenvalue (The ED group);
(A-Double) Abnormal materials with one simple and one double eigenvalue (The SS group);
(A-Triple) Abnormal materials with one triple eigenvalue (The D2 group).

This classification is important because the different types of materials have distinctive expressions of the general solutions for the stress and displacements. An essentially identical classification was reached recently by Ting (1999), by examining four types of eigenvalues. He also made a separate classification for "M_3 materials" whose elastic constants are such that all coefficients of $l_3(\mu)$ vanish.

From the eigenrelation of Eq. (2.11) one can easily show that the **a**- and **b**-vectors associated with an eigenvalue μ are related by the transformation rules

$$\mathbf{a} \equiv \mathbf{Y}^{\mathrm{T}}(\mu)[\beta] \begin{bmatrix} 0 & -\mu^2 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & \mu \\ 0 & \mu & 0 \end{bmatrix} \mathbf{b}, \tag{2.20a}$$

$$\mathbf{b} = -\mathrm{d}\mathbf{E}^{\mathrm{T}}(\mu)/\mathrm{d}\mu[\beta]^{-1}\mathbf{E}(\mu)\mathbf{a}. \tag{2.20b}$$

If $\mathbf{a}^{(j)}$ and $\mathbf{b}^{(j)}$ are the eigenvectors associated with another eigenvalue μ_j, then

$$\mathbf{b}^{(i)\mathrm{T}}\mathbf{a}^{(j)} + \mathbf{a}^{(i)\mathrm{T}}\mathbf{b}^{(j)} = \left\{b_2^{(i)}, b_3^{(i)}\right\}[\![\mu_i, \mu_j]\!]\begin{Bmatrix} b_2^{(j)} \\ b_3^{(j)} \end{Bmatrix} \tag{2.21}$$

where the matrix $[\![\hat{\mu}, \mu]\!]$ is defined by

$$[\![\hat{\mu}, \mu]\!] = \mathbf{J}(\hat{\mu})^{\mathrm{T}}\mathbf{K}(\mu) + \mathbf{K}(\hat{\mu})^{\mathrm{T}}\mathbf{J}(\mu) \tag{2.22}$$

It is straightforwardly verified that

$$[\![\hat{\mu}, \mu]\!] = \frac{1}{\hat{\mu} - \mu}\left\{\mathbf{M}(\hat{\mu}) - \mathbf{M}(\mu)\right\} \quad \text{if } \hat{\mu} \neq \mu \tag{2.23a}$$

whereas

$$[\![\mu, \mu]\!] = \begin{bmatrix} l_4'(\mu) & -l_3'(\mu) \\ -l_3'(\mu) & l_2'(\mu) \end{bmatrix} = \mathbf{M}'(\mu) \tag{2.23b}$$

Eqs. (2.13), (2.21) and (2.23a) imply the orthogonality relation

$$\mathbf{b}^{(i)\mathrm{T}}\mathbf{a}^{(j)} + \mathbf{a}^{(i)\mathrm{T}}\mathbf{b}^{(j)} = 0 \quad \text{if } \mu_i \neq \mu_j \tag{2.24}$$

It is easily seen that the eigenvectors associated with the complex conjugate eigenvalue $\bar{\mu}_j$ are $\bar{\mathbf{a}}^{(j)}$ and $\bar{\mathbf{b}}^{(j)}$. Hence,

$$\mathbf{b}^{(i)\mathrm{T}}\bar{\mathbf{a}}^{(j)} + \mathbf{a}^{(i)\mathrm{T}}\bar{\mathbf{b}}^{(j)} = \left\{b_2^{(i)}, b_3^{(i)}\right\}[\![\mu_i, \bar{\mu}_j]\!]\begin{Bmatrix} \bar{b}_2^{(j)} \\ \bar{b}_3^{(j)} \end{Bmatrix} = 0 \tag{2.25}$$

The last equation is valid for $i = j$ as well.

3. Non-degenerate cases

N-Simple materials and A-Double materials are non-degenerate. There are three pairs of independent eigenvectors $\mathbf{b}^{(i)}$ and $\mathbf{a}^{(i)}$ ($i = 1, 2, 3$) and three pairs of complex conjugate eigenvectors. Eqs. (2.4a) and (2.4b) give the complete representations of the displacements and the stress potentials. The eigenvectors and the Barnett–Lothe tensors are found in the following.

3.1. N-Simple material (the SP group)

For this case $\delta(\mu) = 0$ has three simple roots μ_1, μ_2, and μ_3, with positive imaginary parts. Each eigenvalue has a unique pair of eigenvectors (except for a multiplicative factor) given by

$$\mathbf{b}^{(s)} = \mathbf{J}(\mu_s)\begin{Bmatrix} \sqrt{l_2(\mu_s)} \\ \sqrt{l_4(\mu_s)} \end{Bmatrix} = \begin{Bmatrix} -\mu_s\sqrt{l_2(\mu_s)} \\ \sqrt{l_2(\mu_s)} \\ \sqrt{l_4(\mu_s)} \end{Bmatrix}, \qquad \mathbf{a}^{(s)} = \boldsymbol{K}(\mu_s)\begin{Bmatrix} \sqrt{l_2}(\mu_s) \\ \sqrt{l_4}(\mu_s) \end{Bmatrix} \quad (s = 1, 2, 3) \tag{3.1}$$

Then,

$$\mathbf{b}^{(i)\mathrm{T}}\mathbf{a}^{(j)} + \mathbf{a}^{(i)\mathrm{T}}\mathbf{b}^{(j)} = \left\{ \sqrt{l_2(\mu_i)} \quad \sqrt{l_4(\mu_i)} \right\} [\![\mu_i, \mu_j]\!] \begin{Bmatrix} \sqrt{l_2(\mu_j)} \\ \sqrt{l_4(\mu_j)} \end{Bmatrix} = \left\{ \begin{matrix} 0 & \text{if } i \neq j \\ \delta'(\mu_i) & \text{if } i = j \end{matrix} \right.$$

where $\delta' \equiv l_2 l_4' + l_4 l_2' - 2 l_3 l_3'$. In conjunction with Eq. (2.25), this yields

$$\begin{bmatrix} \mathbf{B}^{\mathrm{T}} & \mathbf{A}^{\mathrm{T}} \\ \bar{\mathbf{B}}^{\mathrm{T}} & \bar{\mathbf{A}}^{\mathrm{T}} \end{bmatrix} \begin{bmatrix} \mathbf{A} & \bar{\mathbf{A}} \\ \mathbf{B} & \bar{\mathbf{B}} \end{bmatrix} = \begin{bmatrix} \boldsymbol{\Omega} & \mathbf{0} \\ \mathbf{0} & \bar{\boldsymbol{\Omega}} \end{bmatrix} \tag{3.2}$$

where $\mathbf{B} \equiv \{\mathbf{b}^{(1)}, \mathbf{b}^{(2)}, \mathbf{b}^{(3)}\}$, $\mathbf{A} \equiv \{\boldsymbol{a}^{(1)}, \boldsymbol{a}^{(2)}, \boldsymbol{a}^{(3)}\}$ and

$$\boldsymbol{\Omega} = \mathbf{B}^{\mathrm{T}}\mathbf{A} + \mathbf{A}^{\mathrm{T}}\mathbf{B} = \begin{bmatrix} \delta'(\mu_1) & 0 & 0 \\ 0 & \delta'(\mu_2) & 0 \\ 0 & 0 & \delta'(\mu_3) \end{bmatrix}$$

After pre- and post-multiplication by appropriate matrices, Eq. (3.2) yields the (modified) closure relations

$$\begin{bmatrix} \mathbf{A} & \bar{\mathbf{A}} \\ \mathbf{B} & \bar{\mathbf{B}} \end{bmatrix} \begin{bmatrix} \boldsymbol{\Omega}^{-1} & \mathbf{0} \\ \mathbf{0} & \bar{\boldsymbol{\Omega}}^{-1} \end{bmatrix} \begin{bmatrix} \mathbf{B}^{\mathrm{T}} & \mathbf{A}^{\mathrm{T}} \\ \bar{\mathbf{B}}^{\mathrm{T}} & \bar{\mathbf{A}}^{\mathrm{T}} \end{bmatrix} = \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix} \tag{3.3}$$

i.e.,

$$\mathrm{Re}\left[2\mathbf{A}\boldsymbol{\Omega}^{-1}\mathbf{B}^{\mathrm{T}} - \mathbf{I}\right] = \mathrm{Re}\left[\mathbf{B}\boldsymbol{\Omega}^{-1}\mathbf{B}^{\mathrm{T}}\right] = \mathrm{Re}\left[\mathbf{A}\boldsymbol{\Omega}^{-1}\mathbf{A}^{\mathrm{T}}\right] = 0$$

Hence, the three matrices

$$\mathbf{L} = 2i\mathbf{B}\boldsymbol{\Omega}^{-1}\mathbf{B}^{\mathrm{T}} \qquad \mathbf{H} = -2i\mathbf{A}\boldsymbol{\Omega}^{-1}\mathbf{A}^{\mathrm{T}}, \qquad \mathbf{S} = -i\left(2\mathbf{A}\boldsymbol{\Omega}^{-1}\mathbf{B}^{\mathrm{T}} - \mathbf{I}\right) \tag{3.4}$$

are all real. They are the Barnett-Lothe tensors. Let $\mathbf{U}(\mu)$ denote the adjoint matrix of $\mathbf{M}(\mu)$, i.e.,

$$\mathbf{U}(\mu) \equiv \begin{bmatrix} l_2(\mu) & l_3(\mu) \\ l_3(\mu) & l_4(\mu) \end{bmatrix} \tag{3.5}$$

Then,

$$\mathbf{M}(\mu)\mathbf{U}(\mu) = \mathbf{U}(\mu)\mathbf{M}(\mu) = \Delta(\mu)\mathbf{I}, \tag{3.6}$$

and

$$\mathbf{L} = 2i\sum_s \{1/\delta'(\mu_s)\}\mathbf{b}^{(s)}\mathbf{b}^{(s)^{\mathrm{T}}}$$

$$= 2i\sum_s \{1/\delta'(\mu_s)\}\boldsymbol{J}(\mu_s)\begin{Bmatrix} \sqrt{l_2}(\mu_s) \\ \sqrt{l_4}(\mu_i) \end{Bmatrix}\left\{\sqrt{l_2}(\mu_s), \sqrt{l_4}(\mu_s)\right\}\mathbf{J}(\mu_s)^{\mathrm{T}}$$

$$= 2i\sum_s \{1/\delta'(\mu_s)\}\mathbf{J}(\mu_s)\mathbf{U}(\mu_s)\mathbf{J}(\mu_s)^{\mathrm{T}}, \tag{3.7a}$$

$$\mathbf{H} = -2i\sum_s \{1/\delta'(\mu_s)\}\mathbf{K}(\mu_s)\mathbf{U}(\mu_s)\mathbf{K}(\mu_s)^{\mathrm{T}}, \tag{3.7b}$$

$$\mathbf{S} = -2i\sum_s \{1/\delta'(\mu_s)\}\mathbf{K}(\mu_s)\mathbf{U}(\mu_s)\mathbf{J}(\mu_s)^{\mathrm{T}} + i\mathbf{I}. \tag{3.7c}$$

Although the eigenvectors which form the matrices **A** and **B** are indeterminate up to multiplicative complex scalar factors, these factors do not appear in the Barnett–Lothe tensors. The latter are uniquely determined by the anisotropic elasticity of the material through the matrices $\mathbf{U}(\mu_s)$, $\mathbf{J}(\mu_s)$ and $\mathbf{K}(\mu_s)$ and the scalars $\delta'(\mu_s)$ ($s = 1, 2, 3$). Eq. (3.7a) was first given by Ting (1997). It will be shown that, in terms of unnormalized eigenvectors, the orthogonality and closure relations (3.2) and (3.3) remain valid for the degenerate and abnormal cases, and consequently for all types of anisotropic materials. Hence, Eq. (3.4) always yields real matrices **L**, **S** and **H**.

3.2. A-Double material (the SS group)

In this case $\delta(\mu) = 0$ has one simple root $\hat{\mu}$ and one double root μ_0 such that $\mathbf{M}(\mu_0) = \mathbf{0}$. Then, $\mathbf{U}(\mu_0) = \mathbf{0}$ and there are two independent **b**-vectors associated with the eigenvalue μ_0, which may be chosen as $\{-\mu_0, 1, 0\}^{\mathrm{T}}$ and $\{0, 0, 1\}^{\mathrm{T}}$. Furthermore, $l_2(\hat{\mu}) \neq 0$ since the quadratic form l_2 has no roots other than μ_0 and $\bar{\mu}_0$. Since l_2 and l_3 have this common pair of complex conjugate roots, one must have

$$l_3(\mu) \equiv \{(\beta_{15}/\beta_{55})\mu - \beta_{24}/\beta_{44}\}l_2(\mu)$$

Hence, a solution of Eq. (2.13) associated with $\hat{\mu}$ is $\{1,\ l_3(\hat{\mu})/l_2(\hat{\mu})\}^{\mathrm{T}} = \{1,\ (\beta_{15}/\beta_{55})\hat{\mu} - \beta_{24}/\beta_{44}\}^{\mathrm{T}}$ and

$$\{\mathbf{b}^{(1)}, \mathbf{b}^{(2)}\} = \mathbf{J}(\mu_0), \qquad \mathbf{b}^{(3)} = \mathbf{J}(\hat{\mu})\begin{Bmatrix} 1 \\ l_3(\hat{\mu})/l_2(\hat{\mu}) \end{Bmatrix} \tag{3.8a}$$

$$\{\mathbf{a}^{(1)}, \mathbf{a}^{(2)}\} = \mathbf{K}(\mu_0), \qquad \mathbf{a}^{(3)} = \mathbf{K}(\hat{\mu})\begin{Bmatrix} 1 \\ l_3(\hat{\mu})/l_2(\hat{\mu}) \end{Bmatrix} \tag{3.8b}$$

Then Eq. (2.25) yields $\mathbf{B}^{\mathrm{T}}\bar{\mathbf{A}} + \mathbf{A}^{\mathrm{T}}\bar{\mathbf{B}} = \mathbf{0}$, and Eqs. (2.21), (2.23a) and (2.23b) yield

$$\mathbf{B}^{\mathrm{T}}\mathbf{A} + \mathbf{A}^{\mathrm{T}}\mathbf{B} = \boldsymbol{\Omega} = \begin{bmatrix} l_4'(\mu_0) & -l_3'(\mu_0) & 0 \\ -l_3'(\mu_0) & l_2'(\mu_0) & 0 \\ 0 & 0 & \delta'(\hat{\mu})/l_2(\hat{\mu}) \end{bmatrix} \tag{3.9a}$$

$$\boldsymbol{\Omega}^{-1} = \begin{bmatrix} 2l_2'(\mu_0)/\delta''(\mu_0) & 2l_3'(\mu_0)/\delta''(\mu_0) & 0 \\ 2l_3'(\mu_0)/\delta''(\mu_0) & 2l_4'(\mu_0)/\delta''(\mu_0) & 0 \\ 0 & 0 & l_2(\hat{\mu})/\delta'(\hat{\mu}) \end{bmatrix} \tag{3.9b}$$

where, using $l_2(\mu_0) = l_3(\mu_0) = l_4(\mu_0) = 0$, one has

$$\delta''(\mu_0) = 2|\mathbf{U}'(\mu_0)| = 2|\mathbf{M}'(\mu_0)| = 2l_2'(\mu_0)l_4'(\mu_0) - 2l_3'(\mu_0)^2.$$

The Barnett–Lothe tensors are obtained by substituting the preceding expression of $\boldsymbol{\Omega}^{-1}$ in Eq. (3.4). The results are

$$\mathbf{L} = \{2i/\delta'(\hat{\mu})\}\mathbf{JUJ}^{\mathrm{T}}(\hat{\mu}) + \{4i/\delta''(\mu_0)\}\mathbf{JU}'\mathbf{J}^{\mathrm{T}}(\mu_0),$$

$$\mathbf{H} = -\{2i/\delta'(\hat{\mu})\}\mathbf{KUK}^{\mathrm{T}}(\hat{\mu}) - \{4i/\delta''(\mu_0)\}\mathbf{KU}'\mathbf{K}^{\mathrm{T}}(\mu_0),$$

$$\mathbf{S} = -\{2i/\delta'(\hat{\mu})\}\mathbf{KUJ}^{\mathrm{T}}(\hat{\mu}) - \{4i/\delta''(\mu_0)\}\mathbf{KU}'\mathbf{J}^{\mathrm{T}}(\mu_0) + i\mathbf{I}. \tag{3.10}$$

4. Degenerate cases (two independent eigenvectors)

In a degenerate (but not extra-degenerate) case, there are only two pairs of independent eigenvectors. Eqs. (2.4a), (2.4b) and (3.1) do not provide the complete representation of the displacements and the stress potentials. Let $\mathbf{b} = \{b_1, b_2, b_3\}$, and $\mathbf{a} = \{a_1, a_2, a_3\}$ be a pair of eigenvectors associated with a *repeated* root μ_0. They satisfy the eigenrelations

$$\mathbf{M}(\mu_0)\begin{Bmatrix} b_2 \\ b_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}, \tag{4.1a}$$

$$\mathbf{b} = \mathbf{J}(\mu_0)\begin{Bmatrix} b_2 \\ b_3 \end{Bmatrix}, \tag{4.1b}$$

$$\mathbf{a} = \boldsymbol{K}(\mu_0)\begin{Bmatrix} b_2 \\ b_3 \end{Bmatrix}. \tag{4.1c}$$

We seek additional solutions of the following form

$$\mathbf{u} = \mathbf{a}^* f(z) + \mathbf{a} y f'(z), \tag{4.2a}$$

$$\mathbf{q} = \mathbf{b}^* f(z) + \mathbf{b} y f'(z), \tag{4.2b}$$

where $z \equiv x + \mu_0 y$, and $\mathbf{b}^* \equiv \{b_1^*, b_2^*, b_3^*\}^{\mathrm{T}}$ and $\mathbf{a}^* \equiv \{a_1^*, a_2^*, a_3^*\}^{\mathrm{T}}$ are vectors to be determined. Then,

$$-\partial x F_{,y} = -b_1^* f'(z) - b_1 y f''(z) = \partial y(-F_{,x}) = \mu_0 b_2^* f'(z) + \mu_0 b_2 y f''(z) + b_2 f'(z)$$

Hence,

$$b_1^* = -\mu_0 b_2^* - b_2 \tag{4.3}$$

The strain and stress components associated with Eqs. (4.2a) and (4.2b) are, respectively,

$$\{\epsilon\} = \mathbf{E}(\mu_0)\left[\mathbf{a}^* f'(z) + \mathbf{a} y f''(z)\right] + \mathbf{E}'(\mu_0)\mathbf{a} f'(z)$$

and

$$\{\sigma\} = \mathbf{P}(\mu_0)\left(\begin{Bmatrix} b_2^* \\ b_3^* \end{Bmatrix} f'(z) + \begin{Bmatrix} b_2 \\ b_3 \end{Bmatrix}\right) y f''(z) + \mathbf{P}'(\mu_0)\begin{Bmatrix} b_2 \\ b_3 \end{Bmatrix} f'(z)$$

Substituting the preceding expressions into the constitutive relation, Eq. (2.1), and using Eqs. (4.1a)–(4.1c), one obtains the eigenrelation for the undetermined vectors $\mathbf{b}^*$ and $\mathbf{a}^*$:

$$\mathbf{E}(\mu_0)\mathbf{a}^* + \mathbf{E}'(\mu_0)\mathbf{a} = [\beta]\mathbf{P}(\mu_0)\begin{Bmatrix} b_2^* \\ b_3^* \end{Bmatrix} + [\beta]\mathbf{P}'(\mu_0)\begin{Bmatrix} b_2 \\ b_3 \end{Bmatrix} \tag{4.4}$$

The equation governing $\{b_2^*, b_3^*\}$ is obtained after premultiplying the last equation and Eq. (4.1c) by the matrices $\mathbf{P}^{\mathrm{T}}(\mu_0)$ and $\mathbf{J}^{\mathrm{T}}(\mu_0)$, respectively, summing the results, and using the derivative of Eq. (2.9b), $\mathbf{P}^{\mathrm{T}}\mathbf{E}' + \mathbf{P}'^{\mathrm{T}}\mathbf{E} = \mathbf{0}$. This yields

$$\mathbf{M}(\mu_0)\begin{Bmatrix} b_2^* \\ b_3^* \end{Bmatrix} + \mathbf{M}'(\mu_0)\begin{Bmatrix} b_2 \\ b_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{4.5}$$

Premultiplying Eq. (4.4) by the matrix $\mathbf{Y}^{\mathrm{T}}(\mu)$, and using Eqs. (2.8a), (2.8b), (2.9a), (2.18b) and (4.1c), one obtains

$$\mathbf{a}^* = \mathbf{K}(\mu_0)\begin{Bmatrix} b_2^* \\ b_3^* \end{Bmatrix} + \mathbf{K}'(\mu_0)\begin{Bmatrix} b_2 \\ b_3 \end{Bmatrix} \tag{4.6}$$

Furthermore, Eqs. (4.1b) and (4.3) yield

$$\mathbf{b}^* = \mathbf{J}(\mu_0)\begin{Bmatrix} b_2^* \\ b_3^* \end{Bmatrix} + \mathbf{J}'(\mu_0)\begin{Bmatrix} b_2 \\ b_3 \end{Bmatrix} \tag{4.7}$$

Notice that Eqs. (4.5)–(4.7), which determine the generalized eigenvectors $\mathbf{b}^*$ and $\mathbf{a}^*$, are different from the eigenrelations ((4.1a)–(4.1c)) governing the eigenvectors $\mathbf{b}$ and $\mathbf{a}$.

The case of a double root (μ_0) with $\mathbf{M}(\mu_0) = \mathbf{0}$ corresponds to A-double materials (the SS Group) discussed in the last section. We next consider N-double materials, which have a double eigenvalue with $\mathbf{M}(\mu_0) \neq \mathbf{0}$. Then $l_2(\mu_0) \neq 0$, for otherwise $l_3(\mu_0)$ and $l_4(\mu_0)$ must also vanish by virtue of $\delta(\mu_0) = \delta'(\mu_0) = 0$. Hence, $\{l_2(\mu_0), l_3(\mu_0)\}^{\mathrm{T}}$ is a nontrivial solution of Eq. (4.1a). Differentiation of Eq. (2.14) yields

$$\mathbf{M}(\mu)\begin{Bmatrix} l_2'(\mu) \\ l_3'(\mu) \end{Bmatrix} + \mathbf{M}'(\mu)\begin{Bmatrix} l_2(\mu) \\ l_3(\mu) \end{Bmatrix} = \delta'(\mu)\begin{Bmatrix} 1 \\ 0 \end{Bmatrix} \tag{4.8}$$

Since $\delta'(\mu_0) = 0$, Eq. (4.5) is satisfied by choosing $\{b_2, b_3\} = \{l_2(\mu_0), l_3(\mu_0)\}$ and $\{b_2^*, b_3^*\} = \{l_2'(\mu_0), l_3'(\mu_0)\}$. Eq. (4.3) then yields $b_1^* = -\mu_0 l_2'(\mu_0) - l_2(\mu_0) = (-\mu l_2)'(\mu_0) = b_1'(\mu_0)$. Consequently,

$$\mathbf{b}^* = \mathrm{d}\mathbf{b}/\mathrm{d}\mu, \qquad \mathbf{a}^* = \mathrm{d}\mathbf{a}/\mathrm{d}\mu, \tag{4.9}$$

where the differentiations are operated on the expressions

$$\mathbf{b} = \mathbf{J}(\mu)\begin{Bmatrix} l_2(\mu) \\ l_3(\mu) \end{Bmatrix}, \qquad \mathbf{a} = \mathbf{K}(\mu)\begin{Bmatrix} l_2(\mu) \\ l_3(\mu) \end{Bmatrix} \tag{4.10}$$

It is understood that the last four expressions will be evaluated at $\mu = \mu_0$ to obtain the eigenvectors and generalized eigenvectors in accordance with the derivative rule. Notice that $\mathbf{b}^*$ is not a null vector because its second component $l_2'(\mu_0) = 2(\beta_{55}\mu_0 - \beta_{45})$ does not vanish (otherwise the eigenvalue $\mu_0 = \beta_{45}/\beta_{55}$ would be real).

A second degenerate case refers to *abnormal* materials with a triple root μ_0. Then, $\mathbf{M}(\mu_0) = \mathbf{0}$ so that Eq. (4.1a) has two independent solutions, one of which may be taken as $\{0, 1\}^{\mathrm{T}}$. Furthermore, $\delta''(\mu_0) = 2\{l_2'(\mu_0)l_4'(\mu_0) - l_3'(\mu_0)^2\} = 0$, but $l_2(\mu_0) \neq 0$ since μ_0 cannot be real. Hence, the second independent solution of Eq. (4.1a) may be chosen as $\{b_2, b_3\}^{\mathrm{T}} = \{l_2'(\mu_0), l_3'(\mu_0)\}^{\mathrm{T}}$. Then, Eq. (4.5) is satisfied by an arbitrary $\{b_2^*, b_3^*\}^{\mathrm{T}}$. We choose $\{b_2^*, b_3^*\}^{\mathrm{T}} = \{l_2''(\mu_0), l_3''(\mu_0)\}^{\mathrm{T}}$ and $b_1^* = (-\mu l_2')'(\mu_0)$ so that Eq. (4.3) and the derivative rule of Eq. (4.9) remain valid, whereas Eq. (4.10) is replaced by

$$\mathbf{b} = \mathbf{J}(\mu)\begin{Bmatrix} l_2'(\mu) \\ l_3'(\mu) \end{Bmatrix}, \qquad \mathbf{a} = \mathbf{K}(\mu)\begin{Bmatrix} l_2'(\mu) \\ l_3'(\mu) \end{Bmatrix}. \tag{4.11}$$

Eqs. (4.2a) and (4.2b) become

$$\mathbf{u} = \mathrm{d}/\mathrm{d}\mu\{\mathbf{a}f(z)\}, \qquad \mathbf{q} = \mathrm{d}/\mathrm{d}\mu\{\mathbf{b}f(z)\}. \tag{4.12}$$

Again, it is understood that all functions of μ associated with a multiple eigenvalue μ_0 are to be evaluated at $\mu = \mu_0$ after performing the required differentiations. This convention will be adopted also in the following analysis. One finds that the derivative rule may also be used to obtain additional eigensolutions associated with generalized eigenvectors.

The Barnett–Lothe tensors for the two degenerate cases are given below.

4.1. N-Double material (the D1 group)

The characteristic equation $\delta(\mu) = 0$ has one double root μ_0 with $l_2(\mu_0) \neq 0$ and a simple root $\hat{\mu}$. The $\mathbf{b}$-vector associated with the simple root $\hat{\mu}$ is chosen to be $\mathbf{b}^{(1)} = \{-\hat{\mu}\sqrt{l_2(\hat{\mu})}, \sqrt{l_2(\hat{\mu})}, \sqrt{l_4(\hat{\mu})}\}^{\mathrm{T}}$. Furthermore, let $\mathbf{b}^{(2)} = -\{\mu l_2(\mu), l_2(\mu), l_3(\mu)\}^{\mathrm{T}}$, and $\mathbf{b}^{(3)} = \mathrm{d}\mathbf{b}^{(2)}/\mathrm{d}\mu$, respectively, be the eigenvector and the generalized eigenvector associated with the double root μ_0. Then,

$$\mathbf{b}^{(1)\mathrm{T}}\mathbf{a}^{(1)} + \mathbf{a}^{(1)\mathrm{T}}\mathbf{b}^{(1)} = \begin{Bmatrix} \sqrt{l_2(\hat{\mu})} & \sqrt{l_4(\hat{\mu})} \end{Bmatrix} [\![\hat{\mu}, \hat{\mu}]\!] \begin{Bmatrix} \sqrt{l_2(\hat{\mu})} \\ \sqrt{l_4(\hat{\mu})} \end{Bmatrix} = \delta'(\hat{\mu}),$$

$$\mathbf{b}^{(2)\mathrm{T}}\mathbf{a}^{(2)} + \mathbf{a}^{(2)\mathrm{T}}\mathbf{b}^{(2)} = \begin{Bmatrix} l_2(\mu), & l_3(\mu) \end{Bmatrix} [\![\mu, \mu]\!] \begin{Bmatrix} l_2(\mu) \\ l_3(\mu) \end{Bmatrix} = l_2(\mu)\delta'(\mu) - l_2'(\mu)\delta(\mu),$$

$$\begin{aligned} \boldsymbol{b}^{(1)\mathrm{T}}\mathbf{a}^{(2)} + \mathbf{a}^{(1)\mathrm{T}}\mathbf{b}^{(2)} &= \begin{Bmatrix} \sqrt{l_2(\hat{\mu})} & \sqrt{l_4(\hat{\mu})} \end{Bmatrix} [\![\hat{\mu}, \mu]\!] \begin{Bmatrix} l_2(\mu) \\ l_3(\mu) \end{Bmatrix} \\ &= \frac{1}{\hat{\mu} - \mu}\left[\frac{l_2(\mu)\sqrt{l_4(\hat{\mu})} + l_3(\mu)\sqrt{l_2(\hat{\mu})}}{\sqrt{l_2(\hat{\mu})}\sqrt{l_4(\hat{\mu})} + l_3(\hat{\mu})}\delta(\hat{\mu}) - \delta(\mu)\sqrt{l_2(\hat{\mu})}\right]. \end{aligned}$$

The last two expressions both vanish since $\delta(\hat{\mu}) = \delta(\mu_0) = \delta'(\mu_0) = 0$. Furthermore, differentiating the last two equations with respect to μ, and using $\delta(\hat{\mu}) = \delta(\mu_0) = \delta'(\mu_0) = 0$, one obtains

$$2\left(\mathbf{b}^{(2)\mathrm{T}}\mathbf{a}^{(3)} + \mathbf{a}^{(2)\mathrm{T}}\mathbf{b}^{(3)}\right) = l_2(\mu_0)\delta''(\mu_0), \qquad \mathbf{b}^{(1)\mathrm{T}}\mathbf{a}^{(3)} + \mathbf{a}^{(1)\mathrm{T}}\mathbf{b}^{(3)} = 0$$

Using Eqs. (4.1b), (4.1c), (4.6) and (4.7), one obtains

$$\mathbf{b}^{(3)\mathrm{T}}\mathbf{a}^{(3)} + \mathbf{a}^{(3)\mathrm{T}}\mathbf{b}^{(3)} =$$

$$\{b_2^*, b_3^*\}\mathbf{M}'(\mu)\begin{Bmatrix} b_2^* \\ b_3^* \end{Bmatrix} + \{b_2^*, b_3^*\}\mathbf{M}''(\mu)\begin{Bmatrix} b_2 \\ b_3 \end{Bmatrix} + 1/6\{b_2, b_3\}\mathbf{M}'''(\mu)\begin{Bmatrix} b_2 \\ b_3 \end{Bmatrix}$$

$$= l_2\delta'''/6 + l_2'\delta''/2 - l_2''\delta'/2.$$

The preceding expressions imply that

$$\mathbf{B}^{\mathrm{T}}\mathbf{A} + \mathbf{A}^{\mathrm{T}}\mathbf{B} = \boldsymbol{\Omega} = \begin{bmatrix} \delta'(\hat{\mu}) & 0 & 0 \\ 0 & 0 & l_2(\mu_0)\delta''(\mu_0)/2 \\ 0 & l_2(\mu_0)\delta''(\mu_0)/2 & l_2(\mu_0)\delta'''(\mu_0)/6 + l_2'(\mu_0)\delta''(\mu_0)/2 \end{bmatrix} \tag{4.13}$$

$$\boldsymbol{\Omega}^{-1} = \begin{bmatrix} 1/\delta'(\hat{\mu}) & 0 & 0 \\ 0 & (2/3)\left(1/\delta''\right)'/l_2(\mu_0) + 2(1/l_2)'/\delta''(\mu_0) & 2/\left\{l_2(\mu_0)\delta''(\mu_0)\right\} \\ 0 & 2/\left\{l_2(\mu_0)\delta''(\mu_0)\right\} & 0 \end{bmatrix} \tag{4.14}$$

For the two pairs of eigenvectors $\{\mathbf{b}^{(1)}, \mathbf{a}^{(1)}\}$ and $\{\mathbf{b}^{(2)}, \mathbf{a}^{(2)}\}$, Eq. (2.25) yields

$$\mathbf{b}^{(1)\mathrm{T}}\bar{\mathbf{a}}^{(1)} + \mathbf{a}^{(1)\mathrm{T}}\bar{\mathbf{b}}^{(1)} = \mathbf{b}^{(1)\mathrm{T}}\bar{\mathbf{a}}^{(2)} + \mathbf{a}^{(1)\mathrm{T}}\bar{\mathbf{b}}^{(2)} = \mathbf{b}^{(2)\mathrm{T}}\bar{\mathbf{a}}^{(2)} + \mathbf{a}^{(2)\mathrm{T}}\bar{\mathbf{b}}^{(2)} = 0.$$

Furthermore, taking the partial derivative of Eq. (2.25) with respect to μ, and using Eqs. (2.23a), (4.1a) and (4.5), one obtains $\mathbf{b}^{(3)\mathrm{T}}\bar{\mathbf{a}}^{(1)} + \mathbf{a}^{(3)\mathrm{T}}\bar{\mathbf{b}}^{(1)} = \mathbf{b}^{(3)\mathrm{T}}\bar{\mathbf{a}}^{(2)} + \mathbf{a}^{(3)\mathrm{T}}\bar{\mathbf{b}}^{(2)} = 0$. Similarly, repeated differentiation of Eq. (2.25) with respect to μ and $\bar{\mu}$ yields $\mathbf{b}^{(3)\mathrm{T}}\bar{\mathbf{a}}^{(3)} + \mathbf{a}^{(3)\mathrm{T}}\bar{\mathbf{b}}^{(3)} = 0$. Thus, the derivative rule implies that the identity

$$\mathbf{B}^{\mathrm{T}}\bar{\mathbf{A}} + \mathbf{A}^{\mathrm{T}}\bar{\mathbf{B}} = \mathbf{0}, \tag{4.15}$$

remains valid when the matrices $\mathbf{B}$ and $\mathbf{A}$ include generalized eigenvectors among the columns. Then, with $\boldsymbol{\Omega}$ and $\boldsymbol{\Omega}^{-1}$ given by Eq. (4.14), the modified orthogonality relations of Eqs. (3.2) and (3.3) are also satisfied. The Barnett–Lothe tensors are

$$\mathbf{L} = 2i\mathbf{B}\boldsymbol{\Omega}^{-1}\mathbf{B}^{\mathrm{T}} = 2i\left[\left\{1/\delta'(\hat{\mu})\right\}\mathbf{JUJ}^{\mathrm{T}}(\hat{\mu}) + \left(2/\delta''\right)\left(\mathbf{JUJ}^{\mathrm{T}}\right)'(\mu_0) + (2/3)\left(1/\delta''\right)'\mathbf{JUJ}^{\mathrm{T}}(\mu_0)\right],$$

$$\mathbf{H} = -2i\mathbf{A}\boldsymbol{\Omega}^{-1}\mathbf{A}^{\mathrm{T}} = -2i\left[\left\{1/\delta'(\hat{\mu})\right\}\mathbf{KUK}^{\mathrm{T}}(\hat{\mu}) + \left(2/\delta''\right)\left(\mathbf{KUK}^{\mathrm{T}}\right)'(\mu_0) + (2/3)\left(1/\delta''\right)'\mathbf{KUK}^{\mathrm{T}}(\mu_0)\right],$$

$$\mathbf{S} = -i(2\mathbf{A}\boldsymbol{\Omega}^{-1}\mathbf{B}^{\mathrm{T}} - \mathbf{I}) = -2i\left[\left\{1/\delta'(\hat{\mu})\right\}\mathbf{KUJ}^{\mathrm{T}}(\hat{\mu}) + \left(2/\delta''\right)\left(\mathbf{KUJ}^{\mathrm{T}}\right)'(\mu_0) + (2/3)\right.$$

$$\left. \times \left(1/\delta''\right)'\mathbf{KUJ}^{\mathrm{T}}(\mu_0)\right] + i\mathbf{I}. \tag{4.16}$$

4.2. A-Triple materials (the D2 group)

$\delta(\mu) = 0$ has a triple root μ_0 with $\mathbf{M}(\mu_0) = \mathbf{0}$. Hence, $\mathbf{U}(\mu_0) = \mathbf{0}$ and

$$\delta''(\mu_0) = 2\left(l_2' l_4' - l_3'^{\,2}\right) = 0, \tag{4.17}$$

where $l_2'(\mu_0) \neq 0$ since $l_2(\mu) = 0$ cannot have a double root. The eigenvectors are

$$\left\{\mathbf{b}^{(1)}, \mathbf{b}^{(2)}\right\} = \mathbf{J}\begin{bmatrix} 0 & l_2' \\ 1 & l_3' \end{bmatrix} \qquad \mathbf{b}^{(3)} = \mathrm{d}\mathbf{b}^{(2)}/\mathrm{d}\mu = \mathbf{J}'\begin{bmatrix} l_2' \\ l_3' \end{bmatrix} + \mathbf{J}\begin{bmatrix} l_2'' \\ l_3'' \end{bmatrix} \tag{4.18}$$

$$\left\{\mathbf{a}^{(1)}, \mathbf{a}^{(2)}\right\} = \mathbf{K}\begin{bmatrix} 0 & l_2' \\ 1 & l_3' \end{bmatrix} \qquad \mathbf{a}^{(3)} = \mathrm{d}\mathbf{a}^{(2)}/\mathrm{d}\mu = \mathbf{K}'\begin{bmatrix} l_2' \\ l_3' \end{bmatrix} + \mathbf{K}\begin{bmatrix} l_2'' \\ l_3'' \end{bmatrix} \tag{4.19}$$

Then,

$$\mathbf{b}^{(1)\mathrm{T}}\mathbf{a}^{(1)} + \mathbf{a}^{(1)\mathrm{T}}\mathbf{b}^{(1)} = \{0, 1\}\left(\mathbf{J}^{\mathrm{T}}\mathbf{K} + \mathbf{K}^{\mathrm{T}}\mathbf{J}\right)\begin{Bmatrix} 0 \\ 1 \end{Bmatrix} = \{0, 1\}\mathbf{M}'\begin{Bmatrix} 0 \\ 1 \end{Bmatrix} = l_2',$$

$$\mathbf{b}^{(1)\mathrm{T}}\mathbf{a}^{(2)} + \mathbf{a}^{(1)\mathrm{T}}\mathbf{b}^{(2)} = \{0, 1\}\mathbf{M}'\begin{Bmatrix} l_2' \\ l_3' \end{Bmatrix} = 0,$$

$$\mathbf{b}^{(1)\mathrm{T}}\mathbf{a}^{(3)} + \mathbf{a}^{(1)\mathrm{T}}\mathbf{b}^{(3)} = \{0, 1\}\left(\mathbf{J}^{\mathrm{T}}\mathbf{K}' + \mathbf{K}^{\mathrm{T}}\mathbf{J}'\right)\begin{Bmatrix} l_2' \\ l_3' \end{Bmatrix} + \{0, 1\}\left(\mathbf{J}^{\mathrm{T}}\mathbf{K} + \mathbf{K}^{\mathrm{T}}\mathbf{J}\right)\begin{Bmatrix} l_2'' \\ l_3'' \end{Bmatrix}$$

$$= \{0, 1\}\left(\mathbf{M}''/2\right)\begin{Bmatrix} l_2' \\ l_3' \end{Bmatrix} + \{0, 1\}\mathbf{M}'\begin{Bmatrix} l_2'' \\ l_3'' \end{Bmatrix} = \left(l_2' l_3'' - l_3' l_2''\right)/2,$$

$$\mathbf{b}^{(3)\mathrm{T}}\mathbf{a}^{(3)} + \mathbf{a}^{(3)\mathrm{T}}\mathbf{b}^{(3)} = \left\{l_2', l_3'\right\}\left(\mathbf{J}'^{\mathrm{T}}\mathbf{K}' + \mathbf{K}'^{\mathrm{T}}\mathbf{J}'\right)\begin{Bmatrix} l_2' \\ l_3' \end{Bmatrix} + 2\left\{l_2'', l_3''\right\}\left(\mathbf{J}^{\mathrm{T}}\mathbf{K}' + \mathbf{K}^{\mathrm{T}}\mathbf{J}'\right)\begin{Bmatrix} l_2' \\ l_3' \end{Bmatrix}$$

$$+\left\{l_2'', l_3''\right\}\left(\mathbf{J}^{\mathrm{T}}\mathbf{K} + \mathbf{K}^{\mathrm{T}}\mathbf{J}\right)\begin{Bmatrix} l_2'' \\ l_3'' \end{Bmatrix}$$

$$= \left\{l_2', l_3'\right\}\left(\mathbf{M}'''/6\right)\begin{Bmatrix} l_2' \\ l_3' \end{Bmatrix} + \left\{l_2'', l_3''\right\}\mathbf{M}''\begin{Bmatrix} l_2' \\ l_3' \end{Bmatrix} + \left\{l_2'', l_3''\right\}\mathbf{M}'\begin{Bmatrix} l_2'' \\ l_3'' \end{Bmatrix}$$

$$= l_2''\delta'''/3 - l_2'\left(l_2'' l_4'' - l_3''^{\,2}\right) + l_2'\delta''''/24,$$

$$\mathbf{b}^{(2)\mathrm{T}}\mathbf{a}^{(2)} + \mathbf{a}^{(2)\mathrm{T}}\mathbf{b}^{(2)} = \left\{l_2', l_3'\right\}\mathbf{M}'\begin{Bmatrix} l_2' \\ l_3' \end{Bmatrix} = l_2'\left(l_2' l_4' - l_3'^{\,2}\right).$$

Differentiation of the last equation yields

$$2\{\mathbf{b}^{(2)\mathrm{T}}\mathbf{a}^{(3)}+\mathbf{a}^{(2)\mathrm{T}}\mathbf{b}^{(3)}\}=l_2'(l_4'l_2''+l_2'l_4''-2l_3'l_3'')=l_2'\delta'''(\mu)/3.$$

Consequently,

$$\boldsymbol{\Omega}=\begin{bmatrix} l_2' & 0 & (l_2'l_3''-l_3'l_2'')/2 \\ 0 & 0 & l_2'\delta'''/6 \\ (l_2'l_3''-l_3'l_2'')/2 & l_2'\delta'''/6 & l_2'\delta''''/24+l_2''\delta'''/3-l_2'(l_2''l_4''-l_3''^2)/4 \end{bmatrix} \tag{4.20a}$$

$$\boldsymbol{\Omega}^{-1}=(3/l_2'\delta''')\begin{bmatrix} \delta'''/3 & -(l_2'l_3''-l_3'l_2'')/l_2' & 0 \\ -(l_2'l_3''-l_3'l_2'')/l_2' & -\delta''''/(2\delta''')-3l_2''/l_2' & 2 \\ 0 & 2 & 0 \end{bmatrix} \tag{4.20b}$$

The derivative rule for deriving the generalized eigenvectors implies the validity of Eq. (4.15) for the present case. The Barnett–Lothe tensors are given by

$$\mathbf{L}=(6i/\delta''')(\mathbf{JUJ}^{\mathrm{T}})''(\mu_0)+3i(1/\delta''')'\mathbf{JU}'\mathbf{J}^{\mathrm{T}}(\mu_0),$$

$$\mathbf{H}=-(6i/\delta''')(\mathbf{KUK}^{\mathrm{T}})''(\mu_0)-3i(1/\delta''')'\mathbf{KU}'\mathbf{K}^{\mathrm{T}}(\mu_0),$$

$$\mathbf{S}=-(6i/\delta''')(\mathbf{KUJ}^{\mathrm{T}})''(\mu_0)-3i(1/\delta''')'\mathbf{KU}'\mathbf{J}^{\mathrm{T}}(\mu_0)+i\mathbf{I}. \tag{4.21}$$

5. Extra-degenerate case (N-Triple materials)

The remaining case is that of N-Triple materials, for which $\delta(\mu)=0$ has a triple root μ_0 with $\mathbf{M}(\mu_0)\neq\mathbf{0}$. Then, none of the elements of $\mathbf{M}(\mu_0)$ can vanish because otherwise μ_0 would be a root of either $l_2l_4=0$ or $l_3^2=0$, and consequently of both (since $\delta(\mu)=l_2l_4-l_3^2=0$), and consequently of $l_2=l_3=l_4=0$ (since μ_0 is a triple root), so that the material would be abnormal.

It follows that the singular matrix $\mathbf{M}(\mu_0)$ is of rank one and Eqs. (4.1a)–(4.1c) yields only one set of independent eigenvectors $\{\mathbf{a},\ \mathbf{b}\}$. The case is extra-degenerate. Eqs. (4.9) and (4.10) give one set of generalized eigenvectors $\{\mathbf{a}^*,\ \mathbf{b}^*\}$, and Eqs. (4.2a) and (4.2b) gives the corresponding eigensolution. We seek an additional independent solution of the form

$$\mathbf{u}=\mathbf{a}^{**}f(z)+\mathbf{a}^*2yf'(z)+\mathbf{a}y^2f''(z),\qquad \mathbf{q}=\mathbf{b}^{**}f(z)+\mathbf{b}^*2yf'(z)+\mathbf{b}y^2f''(z), \tag{5.1}$$

where the vectors $\mathbf{a}$ and $\mathbf{b}$ satisfy Eqs. (4.1a)–(4.1c), and where $\mathbf{b}^*\equiv\{b_1^*,\ b_2^*,\ b_3^*\}^{\mathrm{T}}$, $\mathbf{a}^*\equiv\{a_1^*,\ a_2^*,\ a_3^*\}^{\mathrm{T}}$, $\mathbf{b}^{**}\equiv\{b_1^{**},\ b_2^{**},\ b_3^{**}\}^{\mathrm{T}}$ and $\mathbf{a}^{**}\equiv\{a_1^{**},\ a_2^{**},\ a_3^{**}\}^{\mathrm{T}}$ are vectors to be determined. Then,

$$-\partial_xF_{,y}=-b_1^{**}f'(z)-b_1^*2yf''(z)-b_1y^2f'''(z)$$

$$=\partial_y(-F_{,x})=\mu b_2^{**}f'(z)+\mu b_2^*2yf''(z)+2b_2^*f'(z)+b_22yf''(z)+\mu b_2y^2f'''(z)$$

Hence,

$$b_1^*=-\mu b_2^*-b_2,\qquad b_1^{**}=-\mu b_2^{**}-2b_2^* \tag{5.2}$$

The strain and stress components are give by

$$\{\epsilon\} = \mathbf{E}(\mu)\left[\mathbf{a}^{**}f'(z) + \mathbf{a}^{*}2yf''(z) + \mathbf{a}y^2f'''(z)\right] + \mathbf{E}'(\mu)\left[\mathbf{a}^{*}2f'(z) + \mathbf{a}2yf''(z)\right]. \tag{5.3a}$$

$$\{\sigma\} = \mathbf{P}(\mu)\left\{\begin{matrix} b_2^{**} \\ b_3^{**} \end{matrix}\right\}f'(z) + \left\{\begin{matrix} b_2^{*} \\ b_3^{*} \end{matrix}\right\}2yf''(z) + \left\{\begin{matrix} b_2 \\ b_3 \end{matrix}\right\}y^2f'''(z)$$

$$+\mathbf{P}'(\mu)\left\{\begin{matrix} b_2^{*} \\ b_3^{*} \end{matrix}\right\}2f'(z) + \left\{\begin{matrix} b_2 \\ b_3 \end{matrix}\right\}2yf''(z). \tag{5.3b}$$

Following the procedure of the last section, one obtains, in addition to Eqs. (4.5)–(4.7) for $\mathbf{b}^*$ and $\mathbf{a}^*$, the eigenrelations for $\mathbf{b}^{**}$ and $\mathbf{a}^{**}$:

$$\mathbf{E}(\mu)\mathbf{a}^{**} + 2\mathbf{E}'(\mu)\mathbf{a}^{*} + \mathbf{E}''(\mu)\mathbf{a} = [\beta]\left(\mathbf{P}(\mu)\left\{\begin{matrix} b_2^{**} \\ b_3^{**} \end{matrix}\right\} + 2\mathbf{P}'(\mu)\left\{\begin{matrix} b_2^{*} \\ b_3^{*} \end{matrix}\right\} + \mathbf{P}''(\mu)\left\{\begin{matrix} b_2 \\ b_3 \end{matrix}\right\}\right), \tag{5.4}$$

$$\mathbf{M}(\mu)\left\{\begin{matrix} b_2^{**} \\ b_3^{**} \end{matrix}\right\} + 2\mathbf{M}'(\mu)\left\{\begin{matrix} b_2^{*} \\ b_3^{*} \end{matrix}\right\} + \mathbf{M}''(\mu)\left\{\begin{matrix} b_2 \\ b_3 \end{matrix}\right\} = \left\{\begin{matrix} 0 \\ 0 \end{matrix}\right\}, \tag{5.5}$$

where

$$\mathbf{a}^{**} = \mathbf{K}(\mu)\left\{\begin{matrix} b_2^{**} \\ b_3^{**} \end{matrix}\right\} + 2\mathbf{K}'(\mu)\left\{\begin{matrix} b_2^{*} \\ b_3^{*} \end{matrix}\right\} + \mathbf{K}''(\mu)\left\{\begin{matrix} b_2 \\ b_3 \end{matrix}\right\}, \tag{5.6}$$

$$\mathbf{b}^{**} = \mathbf{J}(\mu)\left\{\begin{matrix} b_2^{**} \\ b_3^{**} \end{matrix}\right\} + 2\mathbf{J}'(\mu)\left\{\begin{matrix} b_2^{*} \\ b_3^{*} \end{matrix}\right\} + \mathbf{J}''(\mu)\left\{\begin{matrix} b_2 \\ b_3 \end{matrix}\right\}. \tag{5.7}$$

Differentiating Eq. (4.8), one obtains

$$\mathbf{M}(\mu)\left\{\begin{matrix} l_2''(\mu) \\ l_3''(\mu) \end{matrix}\right\} + 2\mathbf{M}'(\mu)\left\{\begin{matrix} l_2'(\mu) \\ l_3'(\mu) \end{matrix}\right\} + \mathbf{M}''(\mu)\left\{\begin{matrix} l_2(\mu) \\ l_3(\mu) \end{matrix}\right\} = \left\{\begin{matrix} \delta''(\mu) \\ 0 \end{matrix}\right\}. \tag{5.8}$$

Eqs. (4.5)–(4.7) and (5.5)–(5.7) may be satisfied by choosing

$$\mathbf{b} = \mathbf{J}(\mu)\left\{\begin{matrix} l_2(\mu) \\ l_3(\mu) \end{matrix}\right\}, \qquad \mathbf{a} = \mathbf{K}(\mu)\left\{\begin{matrix} l_2(\mu) \\ l_3(\mu) \end{matrix}\right\} \tag{5.9}$$

$$\mathbf{b}^* = \mathrm{d}\mathbf{b}/\mathrm{d}\mu, \qquad \mathbf{a}^* = \mathrm{d}\mathbf{a}/\mathrm{d}\mu, \tag{5.10}$$

$$\mathbf{b}^{**} = \mathrm{d}^2\mathbf{b}/\mathrm{d}\mu^2, \qquad \mathbf{a}^{**} = \mathrm{d}^2\mathbf{a}/\mathrm{d}\mu^2, \tag{5.11}$$

Notice that $\{b_2^*, b_3^*\} = \{l_2'(\mu_0), l_3'(\mu_0)\}$ and $\{b_2^{**}, b_3^{**}\} = \{l_2''(\mu_0), l_3''(\mu_0)\}$. Furthermore, Eq. (5.1) becomes

$$\mathbf{u} = \mathrm{d}^2/\mathrm{d}\mu^2\{\mathbf{a}f(z)\}, \qquad \mathbf{q} = \mathrm{d}^2/\mathrm{d}\mu^2\{\mathbf{b}f(z)\}. \tag{5.12}$$

Eqs. (5.10)–(5.12) manifest the derivative rule for obtaining $\mathbf{b}^{**}$, $\mathbf{a}^{**}$ and the associated eigensolution. Let

$$\mathbf{B} = \{\mathbf{b}, \mathbf{b}^*, \mathbf{b}^{**}\}, \qquad \mathbf{A} = \{\mathbf{a}, \mathbf{a}^*, \mathbf{a}^{**}\}, \qquad \boldsymbol{\Omega} = \mathbf{B}^{\mathrm{T}}\mathbf{A} + \mathbf{A}^{\mathrm{T}}\mathbf{B}$$

Then, by taking the various derivatives of the identity $\mathbf{b}^{\mathrm{T}}\bar{\mathbf{a}} + \mathbf{a}^{\mathrm{T}}\bar{\mathbf{b}} = \{b_2, b_3\}[\![\mu, \bar{\mu}]\!]\{b_2, b_3\}^{\mathrm{T}}$ with respect to μ and $\bar{\mu}$, using Eq. (2.23a), (4.1a), (4.5), (5.8) and $\delta''(\mu_0) = \delta''(\bar{\mu}_0) = 0$, one obtains $\mathbf{B}^{\mathrm{T}}\bar{\mathbf{A}} + \mathbf{A}^{\mathrm{T}}\bar{\mathbf{B}} = \mathbf{0}$. Furthermore,

$$\boldsymbol{\Omega} = \begin{bmatrix} 0 & 0 & l_2\delta'''/3 \\ 0 & l_2\delta'''/6 & l_2\delta''''/12 + l_2'\delta'''/3 \\ l_2\delta'''/3 & l_2\delta''''/12 + l_2'\delta'''/3 & l_2\delta'''''/30 + l_2'\delta''''/6 + l_2''\delta'''/3 \end{bmatrix} \tag{5.13a}$$

$$\boldsymbol{\Omega}^{-1} = \frac{3}{l_2 d'''}\begin{bmatrix} \dfrac{-l_2''}{l_2} + 2\left(\dfrac{l_2'}{l_2}\right)^2 + \dfrac{1}{8}\left(\dfrac{\delta''''}{\delta'''}\right)^2 + \dfrac{l_2'\delta''''}{2l_2\delta'''} - \dfrac{\delta'''''}{10\delta'''} & -\left(\dfrac{2l_2'}{l_2} + \dfrac{\delta''''}{2\delta'''}\right) & 1 \\ -\left(\dfrac{2l_2'}{l_2} + \dfrac{\delta''''}{2\delta'''}\right) & 2 & 0 \\ 1 & 0 & 0 \end{bmatrix} \tag{5.13b}$$

The Barnett–Lothe tensors are

$$\mathbf{L} = (6i/\delta''')(\mathbf{JUJ}^{\mathrm{T}})'' + 3i(1/\delta''')'(\mathbf{JUJ}^{\mathrm{T}})' + (6i/19)(1/\delta''')''\mathbf{JUJ}^{\mathrm{T}},$$

$$\mathbf{H} = -(6i/\delta''')(\mathbf{KUK}^{\mathrm{T}})'' - 3i(1/\delta''')'(\mathbf{KUK}^{\mathrm{T}})' - (6i/19)(1/\delta''')''\mathbf{KUK}^{\mathrm{T}},$$

$$\mathbf{S} = -(6i/\delta''')(\mathbf{KUJ}^{\mathrm{T}})'' - 3i(1/\delta''')'(\mathbf{KUJ}^{\mathrm{T}})' - (6i/19)(1/\delta''')''\mathbf{KUJ}^{\mathrm{T}} + i\mathrm{I} \tag{5.14}$$

Notice that Eq. (5.14) for N-Triple materials reduces to Eq. (4.21) for A-Triple materials as $\mathbf{U}(\mu_0)$ vanishes. Likewise, Eq. (4.16) for N-Double materials reduces to Eq. (3.10) for A-Double materials as $\mathbf{U}(\mu_0)$ vanishes. Therefore, the expressions of the Barnet–Lothe tensors of normal materials remain valid for abnormal materials having the same multiplicity of eigenvalues. They are given by Eqs. (3.7a)–(3.7c), (4.16) and (5.14), respectively, when the multiplicity is 1, 2 or 3. These expressions show that each Barnett–Lothe tensor is composed of separate terms associated with the various eigenvalues. A simple root of $\delta(\mu) = 0$ contributes a term $(2i/\delta')\,\mathbf{JUJ}^{\mathrm{T}}$ to the tensor $\mathbf{L}$, a double root contributes $(4i/\delta'') \times (\mathbf{JUJ}^{\mathrm{T}})' + (4i/3)(1/\delta'')'\mathbf{JUJ}^{\mathrm{T}}$, and a triple root gives the right-hand side of the first equation of Eq. (5.14).

6. General solutions for the displacements, the stress and the strain

In the two non-degenerate cases, complete solutions of the displacements and stress resultants are given by Eqs. (2.4a) and (2.4b), where the eigenvectors are given by Eq. (3.1) for N-simple materials (the SP Group) and by Eqs. (3.8a) and (3.8b) for A-Double materials (the SS group). In the two degenerate cases (N-Double and A-Triple materials), the number of conjugate pairs of independent solutions is reduced by one due to the corresponding reduction in the number of independent eigenvectors, and an additional solution is provided by Eqs. (4.2a), (4.2b), (4.9) and (4.10) for N-Double materials and by Eqs. (4.2a), (4.2b), (4.9) and (4.11) for A-Triple materials. In the extra-degenerate case (N-Triple materials), Eq. (5.9) and $\mathbf{u} = \mathbf{a}f(x + \mu_0 y)$, $\mathbf{q} = \mathbf{b}f(x + \mu_0 y)$ provide one independent eigensolution. An additional solution is given by Eqs. (4.2a), (4.2b), (4.9) and (5.9). A third eigensolution is provided by

Eqs. (5.1) and (5.9)–(5.11). In all cases, the complex conjugates of the eigenvectors or generalized eigenvectors are also eigenvectors or generalized eigenvectors associated with the conjugate eigenvalue. For **u** and **q** to be real-valued, one must have $f_{i+3}(x+\mu y)=f_i(x+\bar{\mu}y)$.

For the five distinct cases, explicit forms of the general solutions for the displacements, the stress and the strain are shown in the following in terms of the matrices **J**, **P**, **E** and **K** defined by Eqs. (2.2a), (2.8a), (2.8b), (2.18a) and (2.18b).

6.1. N-Simple materials (the SP group)

$\delta(\mu)=0$ has three distinct pairs of complex conjugate roots

$$\mathbf{u}=\sum_{i=1}^{3}\mathrm{Re}\left[f_i(x+\mu_i y)\mathbf{K}(\mu_i)\begin{Bmatrix}\sqrt{l_2(\mu_i)}\\ \sqrt{l_4(\mu_i)}\end{Bmatrix}\right],\qquad \mathbf{q}=\sum\mathrm{Re}\left[f_i(x+\mu_i y)\mathbf{J}(\mu_i)\begin{Bmatrix}\sqrt{l_2(\mu_i)}\\ \sqrt{l_4(\mu_i)}\end{Bmatrix}\right],$$

$$\{\sigma\}=\sum\mathrm{Re}\left[f_i'(x+\mu_i y)\mathbf{P}(\mu_i)\begin{Bmatrix}\sqrt{l_2(\mu_i)}\\ \sqrt{l_4(\mu_i)}\end{Bmatrix}\right],\qquad \{\epsilon\}=\sum\mathrm{Re}\left[f_i'(x+\mu_i y)\mathbf{E}(\mu_i)\mathbf{K}(\mu_i)\begin{Bmatrix}\sqrt{l_2(\mu_i)}\\ \sqrt{l_4(\mu_i)}\end{Bmatrix}\right].$$

6.2. A-Double materials (the SS group)

One double root μ_0 with $\mathbf{M}(\mu_0)=\mathbf{0}$ and one simple root $\hat{\mu}$

$$\mathbf{u}=\mathrm{Re}\left[\mathbf{K}(\mu_0)\begin{Bmatrix}f_1(x+\mu_0 y)\\ f_2(x+\mu_0 y)\end{Bmatrix}+f_3(x+\hat{\mu}y)\mathbf{K}(\hat{\mu})\begin{Bmatrix}l_2(\hat{\mu})\\ l_3(\hat{\mu})\end{Bmatrix}\right].$$

To obtain **q**, replace the **K** matrices in the preceding expression by **J**. To obtain $\{\sigma\}$ and $\{\epsilon\}$, replace **K** by **P** and **EK**, respectively, and then replace the functions f_i by the derivatives f_i'. The same rule also applies to the following three degenerate and extra-degenerate cases.

6.3. N-Double materials (the D1 group)

One double root μ_0 with $l_2(\mu_0)\neq 0$ and one simple root $\hat{\mu}$

$$\mathbf{u}=\mathrm{Re}\left[f_1(x+\mu y)\mathbf{K}(\mu)\begin{Bmatrix}l_2(\mu)\\ l_3(\mu)\end{Bmatrix}+\frac{\mathrm{d}}{\mathrm{d}\mu}\left(f_2(x+\mu y)\mathbf{K}(\mu)\begin{Bmatrix}l_2(\mu)\\ l_3(\mu)\end{Bmatrix}\right)+f_3(x+\hat{\mu}y)\mathbf{K}(\hat{\mu})\begin{Bmatrix}\sqrt{l_2(\hat{\mu})}\\ \sqrt{l_4(\hat{\mu})}\end{Bmatrix}\right].$$

6.4. A-Triple materials (the D2 group)

One triple root μ_0 with $l_2(\mu_0)=l_3(\mu_0)=l_4(\mu_0)=0$

$$\mathbf{u}=\mathrm{Re}\left[f_1(x+\mu y)\mathbf{K}(\mu)\begin{Bmatrix}0\\ 1\end{Bmatrix}+f_2(x+\mu y)\mathbf{K}(\mu)\begin{Bmatrix}l_2'(\mu)\\ l_3'(\mu)\end{Bmatrix}+\frac{\mathrm{d}}{\mathrm{d}\mu}\left(f_3(x+\mu y)\mathbf{K}(\mu)\begin{Bmatrix}l_2'(\mu)\\ l_3'(\mu)\end{Bmatrix}\right)\right]$$

6.5. N-Triple materials (the ED group)

One triple root μ_0 with nonvanishing $l_2(\mu_0)$, $l_3(\mu_0)$ and $l_4(\mu_0)$

$$\mathbf{u} = \mathrm{Re}\left[f_1(x+\mu y)\mathbf{K}(\mu)\begin{Bmatrix} l_2(\mu) \\ l_3(\mu) \end{Bmatrix} + \frac{\mathrm{d}}{\mathrm{d}\mu}\left(f_2(x+\mu y)\mathbf{K}(\mu)\begin{Bmatrix} l_2(\mu) \\ l_3(\mu) \end{Bmatrix}\right) + \frac{\mathrm{d}^2}{\mathrm{d}\mu^2}\left(f_3(x+\mu y)\mathbf{K}(\mu)\begin{Bmatrix} l_2(\mu) \\ l_3(\mu) \end{Bmatrix}\right)\right]$$

In accordance with the derivative rule, all functions of μ in the expressions of the last three material types are to be evaluated at $\mu = \mu_0$ *after* performing the required differentiations.

Notice that for N-Simple and N-Double materials, the vector $\{\sqrt{l_2(\mu)}, \sqrt{l_2(\mu)}\}^{\mathrm{T}}$ associated with a simple eigenvalue μ may be replaced either by $\{l_2(\mu), l_3(\mu)\}^{\mathrm{T}}$ or by $\{l_3(\mu), l_4(\mu)\}^{\mathrm{T}}$, since the last two vectors do not both vanish for a simple eigenvalue. Therefore, in all cases, the eigenvectors, generalized eigenvectors and the general solutions for the displacements and stresses may be chosen to be *polynomial* functions of the associated eigenvalues.

7. Summary

Regarding the nature of general solutions under generalized plane deformation, anisotropic linearly elastic materials may be classified into five mutually exclusive types depending on the multiplicity of eigenvalues and, in cases of a repeated eigenvalue $\mu = \mu_0$, whether the eigenmatrix $\mathbf{M}(\mu)$ of Eq. (2.2b) vanishes at $\mu = \mu_0$. These five classes of materials have different representations of the eigenvectors and of the general solutions for the displacements, the stress and the strain, as given in the last section. In the two non-degenerate cases (N-Simple and A-Double materials), there are three independent pairs of **a**- and **b**-vectors determined by the eigenrelations (2.11)–(2.13). In the two degenerate cases (N-Double and A-Triple materials), an additional eigensolution in the form of Eqs. (4.2a) and (4.2b) may be found where the generalized eigenvectors $\mathbf{a}^*$ and $\mathbf{b}^*$ satisfy a different set of eigenrelations. These relations are given by Eqs. (4.4) and (4.5), instead of Eqs. (2.11) and (2.13). In the extra-degenerate case (N-Triple materials), Eq. (5.1) gives a second generalized eigensolution involving a second pair of generalized eigenvectors $\mathbf{a}^{**}$ and $\mathbf{b}^{**}$. They satisfy the eigenrelations of Eqs. (5.4) and (5.5). The eigenrelations governing the generalized eigenvectors imply that the latter may be computed from suitably chosen *unnormalized* eigenvectors according to the derivative rule (Eqs. (4.9) and (5.10)–(5.11)). The derivative rule also applies to the eigensolutions of the displacements and the stress potentials. The implementation of the derivative rule is facilitated by using *unnormalized* eigenvectors whose components are polynomial functions of the eigenvalues, viz., Eq. (4.10) for normal materials and Eq. (4.11) for abnormal materials.

The unnormalized **a**- and **b**-vectors satisfy the modified orthogonality and closure relations (Eq. (3.2) and (3.3), respectively). These relations contain a matrix $\boldsymbol{\Omega}$ depending on the type of the material and the choice of **a**- and **b**-vectors. A significant amount of algebraic manipulation is required to obtain $\boldsymbol{\Omega}$ in each case, but the inverse of $\boldsymbol{\Omega}$ is easily obtained in closed form. The closure relation implies the expressions of the real matrices **L**, **S** and **H** as given by Eq. (3.4). Notice that these expressions, as well as the modified orthogonality and closure relations of Eqs. (3.2)–(3.4), have identical forms for all five types of anisotropic elastic materials.

By substituting the matrices **B** and **A** into $\boldsymbol{\Omega} = \mathbf{B}^{\mathrm{T}}\mathbf{A} + \mathbf{A}^{\mathrm{T}}\mathbf{B}$, we obtain explicit analytical expressions of the Barnett–Lothe tensors for all types of normal and abnormal anisotropic materials, i.e., Eqs. (3.7a)–(3.7c), (4.16) and (5.14), respectively, when the characteristic equation has single, double and triple roots. The simple matrix forms of these expressions make them exceedingly easy to evaluate for a specific material by using computer algebra. Expressions equivalent to Eqs. (3.7a)–(3.7c) were given recently by Ting (1997) for the case of three distinct eigenvalues. These expressions are not valid for a repeated eigenvalue μ because the terms associated with μ contain the factor $1/\delta'(\mu)$. Ting and Lee (1997) found that the terms may be partitioned and formally rearranged, resulting in alternative expressions that remain bounded for the repeated eigenvalue.

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